On the algebraic geometry of Kac-Moody groups (Topological Field Theory and Related Topics)

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On the algebraic geometry of Kac–Moody groups

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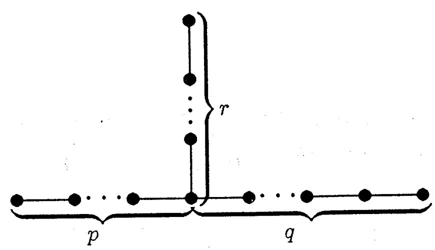
These notes are a slightly elaborated version of a talk given at the RIMS–Symposium on "Topological Field Theory and Related Topics", Kyoto, December 1996. Their aim is to give a survey of the main results obtained by Claus Mokler in his dissertation at Hamburg University ([8], October 1996) pertaining to a natural semigroup completion of Kac–Moody groups.

1. "Abstract" Kac–Moody groups

Starting point for the construction of Kac–Moody Lie algebras and associated groups is a generalized Cartan matrix, i.e. an $l \times l$-matrix $A = ((a_{ij})) \in M_l(\mathbb{Z})$ satisfying

$$\begin{array}{rcll} a_{ii} & = & 2 & \\ a_{ij} & \leq & 0 & i \neq j \\ a_{ij} & = & 0 \Rightarrow & a_{ji} = 0 \end{array}$$

We shall assume, in addition, that A is symmetrizable (cf. [2]). In fact, one might take A to be symmetric for simplicity. Also, the generalized Cartan matrices arising in singularity theory and providing the original motivation for our research in Kac–Moody groups (cf. [11], [13]) are symmetric, e.g. the matrix of type T_{pqr} encoded by the Coxeter–Dynkin diagram



Whereas the Kac–Moody algebra $\mathbf{g} = \mathbf{g}(A)$ is essentially generated by l copies of the Lie algebra $\mathbf{sl}_2(\mathbb{C})$,

$$\langle e_i, h_i, f_i \rangle \ , = i = 1, \dots, l\,,$$

subject to relations derived from A, the corresponding Kac–Moody group $G = G(A)$ is essentially generated by l copies of the Lie group $SL_2(\mathbb{C})$. Here, the relations are either imposed abstractly (Tits, cf.[15], [16]) or by the "integration" of G from the integrable representations of $\mathbf{g}$ (Moody–Teo, Marcuson, Garland, and, in the most thorough way, Kac–Peterson [10], [3], [4]).

The most important result about G as an abstract group is the existence of a "twin" BN–pair or "twin" Tits system (B^+, B^-, N, S) in G providing us, among others, with

- *positive* and *negative Borel subgroups* B^+ and B^-,
- a *maximal torus* $T = B^+ \cap B^- = N \cap B^+ = N \cap B^-$,
- a *Weyl group* $W = N/T$ with generating set S,
- *Bruhat decompositions*

  $$G = \bigcup_{w \in W} B^+ w B^+ = \bigcup_{w \in W} B^- w B^-,$$

  and a *Birkhoff–decomposition*

  $$G = \bigcup_{w \in W} B^- w B^+\,.$$

Similarly, as in the case of the Lie algebra $\mathbf{g}$ where one usually adjoins additional derivations to a "minimal" Kac–Moody algebra, the precise structure of G depends on slightly finer data than A. These data are given by an *integral realization* $(H, \Pi, \Pi^\check{})$ of A which fixes the size of the maximal torus T and its position inside G.

Here, H is the lattice of algebraic one–parameter subgroups $\mathbb{C}^* \to T$ into T with dual $P = H^* = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$, the lattice of algebraic characters $T \to \mathbb{C}^*$, and

$$\Pi = \{\alpha_1, \dots, \alpha_e\} \subset P,\ \Pi^\check{} = \{h_1, \dots, h_l\} \subset H$$

are free subsets of *simple roots* in P, resp. of *simple coroots* in H, related by

$$\alpha_i(h_j) = a_{ij}.$$

More explicitly, Π and $\Pi^\vee$ are given in our context as follows:

Let $\kappa_i : SL_2(\mathbb{C}) \to G$, $i = 1, \ldots, l$ denote the basic homomorphisms of $SL_2(\mathbb{C})$ into G, and let

$$\begin{array}{lcl} h_i : \mathbb{C}^* & \to & G \\ u_i : \mathbb{C} & \to & G \end{array}$$

be given by

$$h_i(s) \; := \; \kappa_i(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}) \quad , \quad s \in \mathbb{C}^*,$$

$$u_i(c) \; := \; \kappa_i(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}) \quad , \quad c \in \mathbb{C},$$

Then $h_i(\mathbb{C}^*) \subset T$, i.e. $h_i \in H$, and there is a character $\alpha_i \in P$ such that

$$t\, u_i(c) t^{-1} = u_i(\alpha_i(t)c)$$

for all $t \in T$, $c \in \mathbb{C}$.

By its natural action on T and P, the Weyl group $W = N/T$ is identified with the subgroup of $\mathrm{Aut}_{\mathbb{Z}}(P)$ generated by the reflections $S = \{s_1, \ldots, s_l\}$

$$s_i(\omega) = \omega - \omega(h_i)\alpha_i \;,\; \omega \in P.$$

Also, s_i is given by the class of

$$\kappa_i(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \text{ in } N/T.$$

We can also make the groups B^+ and B^- more explicit:

Let U_i denote the subgroup $u_i(\mathbb{C})$ and, for any *real root* $\gamma = w(\alpha_i)$ $(w \in W)$, put

$$U_\gamma := w\, U_i w^{-1}.$$

The set $\sum^{\text{real}} = W(\Pi)$ of all real roots divides naturally into positive and negative roots,

$$\sum{}^{\text{real}} = \sum{}^{\text{real},+} \cup \sum{}^{\text{real},-}$$

where $\sum{}^{\text{real},-} = -\sum{}^{\text{real},+}$, and if we put

$$U^{\pm} = \langle U_\gamma | \gamma \in \sum{}^{\text{real},\pm} \rangle$$

($\langle a, b, \ldots \rangle$ denoting the group generated by $a, b, \ldots$) we have

$$B^+ = T \ltimes U^+ \ , \ B^- = T \ltimes U^- \, .$$

Finally, the anti-involution

$$\begin{array}{ccc} SL_2(\mathbb{C}) & \longrightarrow & SL_2(\mathbb{C}) \\ g & \longmapsto & {}^t g \end{array}$$

can be lifted to all of G, i.e. there is an anti-involution $* : G \to G$ such that

- $*(t) = t$, for all $t \in T$
- $*(\kappa_i(g)) = \kappa_i({}^t g)$, for all $g \in SL_2(\mathbb{C})$.

In particular, one has $*(U^+) = U^-, *(U^-) = U^+$.

2. "Algebraic" Kac–Moody groups

If A is a Cartan matrix of "finite type" (i.e. all components are of type $A_n, B_n, \ldots$, F_n, or G_2) then G, as described in the last section, is a reductive algebraic group over $\mathbb{C}$. The algebra $\mathbb{C}[G]$ of regular functions on G is then a Hopf algebra, and the group G can be completely recovered from the Hopf algebra $\mathbb{C}[G]$, in particular

$$G = \text{Specmax}\, \mathbb{C}[G] = \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[G], \mathbb{C}).$$

If A is a proper generalized Cartan matrix, then the associated algebra $\mathbf{g}$ is of infinite dimension over $\mathbb{C}$. Thus, also G should be infinite-dimensional. A proposal for an algebra of "strongly regular" functions on G was made by Kac and Peterson in 1983 ([3]). As in the finite-dimensional case, this algebra is generated by the matrix coefficients of a suitable representation. Let us therefore recall some basic facts about the irreducible highest weight representations of G.

To simplify the presentation, we shall assume that G is of "simply–connected type", i.e. that the coroot lattice $Q^\check{} = \mathbb{Z}.\Pi^\check{}$ is a direct summand of H

$$H = Q^\check{} \oplus D.$$

Then the set $P^+ = \{\omega \in P | \omega(h_i) \geq 0, i = 1, \ldots, l\}$ of *dominant weights* can be written as a direct sum

$$P^+ = P^0 \oplus \bigoplus_{i=1}^{l} \mathbb{N}.\Lambda_i$$

where

$$P^0 = \{\omega \in P | \omega(h_i) = 0, i = 1, \ldots, l\} \cong D^*$$

and where $\Lambda_i, i = 1, \ldots, l$, are *fundamental dominant weights*

$$\Lambda_i(h_j) = \delta_{ij} \ , i, j = 1, \ldots, l,$$

uniquely determined modulo P^0.

As in the finite–dimensional case there is a bijection of P^+ onto the set of isomorphism classes of irreducible highest weight representations L of G

$$\Lambda \in P^+ \longleftrightarrow L(\Lambda)$$

determined by $L(\Lambda)$ having a unique (up to scalars) highest weight vector $v_\Lambda \in L(\Lambda) \backslash \{0\}$ of weight Λ. (If $\Lambda \in P^0$, the module $L(\Lambda)$ will be one–dimensional.)

Any such module carries a nondegenerate contravariant form (essentially unique), i.e. a symmetric bilinear form

$$\langle \ \ , \ \ \rangle : L(\Lambda) \times L(\Lambda) \to \mathbb{C}$$

such that $\langle v, gw \rangle = \langle g^* v, w \rangle$ for all $v, w \in L(\Lambda), g \in G$, and $g^* = *(g)$ the anti–involution on G.

Let us call the function

$$c_{v,w} : G \to \mathbb{C}$$

given by $c_{v,w}(g) = \langle v, gw \rangle$ for some $v, w \in L(\Lambda)$ a *matrix coefficient* of G (in the respresentation $L(\Lambda)$). Kac and Peterson now define

$$\mathbb{C}[G] := \begin{pmatrix} \mathbb{C}\text{-algebra generated by} \\ \text{the matrix coefficients} \\ c_{v,w} \text{ for all } v, w \in L(\Lambda) \\ \text{and all } \Lambda \in P^+ \end{pmatrix}$$

and they prove the following

"Peter–Weyl"–Theorem: The map

$$\bigoplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda) \to \mathbb{C}[G]$$

induced by $v \otimes w \longmapsto c_{v,w}$ is an isomorphism of $G \times G$–modules.

Here, the action of $G \times G$ on $\mathbb{C}[G]$ is given by $((g,h)f)(x) = f(g^* x h)$. Alternatively, one might use the usual action of $G \times G$ on functions on G and let act G on the first factor $L(\Lambda)$ by the contragredient action

$$(g, v) \longmapsto (g^*)^{-1} v.$$

It turned out that $\mathbb{C}[G]$ is not a Hopf algebra. There is neither a co–multiplication nor an antipode (basically due to the infinite–dimenionality of the $L(\Lambda)$ and the inequivalence between highest weight and lowest weight representations). Even worse, Kac and Peterson exhibited elements in Specmax $\mathbb{C}[G]$ not contained in G (which injects into Specmax $\mathbb{C}[G]$) (cf. [3] Remark 2.2). Thus they formulated the following problem (loc. cit., 4H b)):

Determine Specmax $\mathbb{C}[G]$ (possibly with respect to a topological structure on the algebra $\mathbb{C}[G]$)!

Inspired by the deformation theory of certain singularities (cf. [13]) we conjectured

$$\overline{G} := \operatorname{Specmax} \mathbb{C}[G] = G.\overline{T}.G$$

where $\overline{T}$ is the closure of T in $\overline{G}$ realized as the torus embedding

$$T = \operatorname{Specmax} \mathbb{C}[P] \subset \operatorname{Specmax} \mathbb{C}[P \cap I] = \overline{T}$$

for $I \subset P \otimes_{\mathbb{Z}} \mathbb{R}$, the Tits cone attached to G. This embedding, or rather a domain $\mathcal{T} \subset \overline{\mathcal{T}}$ of discontinuity for the action of W, had been studied before by Looijenga and the quotient $\overline{\mathcal{T}}/W$ had turned out to be the base space of a semiuniversal deformation for certain isolated singularities (cf. [6], [7]). Moreover, in [12], [13] we realized $\overline{\mathcal{T}}/W$ and $\overline{T}/W$ as target spaces for an adjoint quotient of G.

During a stay at MSRI (1984), D. Peterson announced a proof of the above conjecture including a number of structural properties of $\overline{G}$ ([9], $\overline{G}$ being considered as the continuous spectrum with respect to some topology). In connection with his infinite-dimensional algebraic-geometric approach to the flag manifolds of Kac–Moody groups, M. Kashiwara also studied the abstract maximal spectrum of $\mathbb{C}[G]$ (without topology on $\mathbb{C}[G]$), cf. [5]. Finally, C. Mokler ([8]) made a quite thorough study of $\overline{G}$ in the context of some infinite-dimensional algebraic geometry based on suitably topologized coordinate rings. In particular, he gave a detailed proof of our conjecture. This is what we want to report upon.

3. A topology on the algebra of strongly regular functions

Let V be a complex vector space. Then we may view the symmetric algebra $S(V^*)$ of its dual space V^* as the coordinate ring of the variety V. If $\dim_{\mathbb{C}} V < \infty$ we have

$$\mathrm{Hom}_{k-\mathrm{alg}}(S(V^*), \mathbb{C}) = \mathrm{Hom}\,(V^*, \mathbb{C}) = V^{**} = V.$$

However, if $\dim_{\mathbb{C}} V = \infty$ we have $V \subset V^{**}, V \neq V^{*,*}$, and $\mathrm{Specmax}\, S(V^*)$ is strictly larger than V. To remedy this defect we put the following topology on the algebra $S(V^*)$:

A basis of neighborhoods of $0 \in S^*(V^*)$ is given by the "cofinite" ideals

$$\{J(V')|V' \subset V \text{ a finite-dimensional subspace}\},$$

$$J(V') = \{f \in S(V^*)|f|_{V'} \equiv 0\}\,.$$

Now, the continuous maximal spectrum

$$\mathrm{Specm}^\circ S(V^*) = \mathrm{Hom}_{\mathrm{cont}-k-\mathrm{alg}}(S(V^*), \mathbb{C})$$

is easily identified with V (i.e. Hilbert's Nullstellensatz gives $V' = S(V^*)/J(V')$ for all finite-dimensional $V' \subset V$).

To put a topology on $\mathbb{C}[G]$ we embed G, and finally $\overline{G}$, into a larger space M constructed as follows:

We fix contravariant forms $\langle \ , \ \rangle$ on all modules $L(\Lambda), \Lambda \in P^+$, and extend them to a form, also denoted by $\langle \ , \ \rangle$, on the direct sum

$$L := \bigoplus_{\Lambda \in P^+} L(\Lambda)$$

by requiring $L(\Lambda)$ and $L(\Lambda')$ to be orthogonal for $\Lambda \neq \Lambda'$. Let M denote the subalgebra of End (L) satisfying

- $\varphi(L(\Lambda)) \subset L(\Lambda)$ for all $\Lambda \in P^+$,
- the adjoint φ^* of φ with respect to $\langle \ , \ \rangle$ exists.

We let $\mathbb{C}[M]$ denote the $\mathbb{C}$–algebra generated by all matrix coefficients $c_{v,w} : M \to \mathbb{C}$, $v, w \in L$, $c_{v,w}(\varphi) = \langle v, \varphi w \rangle$, and consider the "cofinite" topology on $\mathbb{C}[M]$ given by the neighborhood basis of 0

$$\{J(M') | M' \subset M \text{ a subspace of finite dimension}\},$$

$J(M')$ being the vanishing ideal of M'.

Then we have

- $\text{Specm}^\circ \mathbb{C}[M] = M$
- M is a "weak" algebraic monoid (i.e. right and left multiplication on M by given elements of M are "morphisms" of M; note that there is no comultiplication on $\mathbb{C}[M]$).

By the definition of the contravariant forms on the $L(\Lambda)$ and L we have a natural embedding $G \hookrightarrow M$. Moreover, $\mathbb{C}[G]$ is the image of $\mathbb{C}[M]$ under the restriction from M to G. We now put the quotient topology with respect to $\mathbb{C}[M] \to \mathbb{C}[G]$ on $\mathbb{C}[G]$ and we obtain

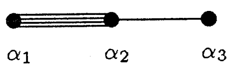
- $\mathrm{Specm}^\circ \mathbb{C}[G] = \overline{G}$ = Zariski–closure of G in M,
- $\overline{G}$ is a "weak" algebraic monoid (in the sense above).

4. The Tits cone and the closure of the maximal torus

Let $V = P \otimes_{\mathbb{Z}} \mathbb{R}$ be the "real" character group, $\overline{C} = \{\omega \in V | \omega(h_i) \geq 0$ for all $i = 1, \ldots, l\}$ a *fundamental Weyl chamber*, and $I = W.\overline{C}$ the union of all W–translates of $\overline{C}$. Then I is a convex solid cone, called the *Tits cone.* The interior I° of I is a domain of discontinuity of W. (For details, cf. [2]).

Example: Let A be the "hyperbolic" matrix

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{corresponding to}$$



.

Now, the matrix A defines a symmetric bilinear form on $V \cong \mathbb{R}^3$ of signature $(+, +, -)$, and with respect to some convention I° may be be identified with the interior of the positive light cone. The Weyl group W is isomorphic to $PGL_2(\mathbb{Z})$ acting as a group of hyperbolic motions on the unit disc $\cong \mathbb{P}(I^\circ) \subset \mathbb{P}(V)$.

The boundary of I is of particular interest for us. A subset $I' \subset I$ is called a (*rational*) *boundary component* of I if there is a $\gamma \in V^* = H \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. a $\gamma \in H$) such that

- $\omega(\gamma) \geq 0$ for all $\omega \in I$
- $\omega(\gamma) = 0$ for $\omega \in I$ implies $\omega \in I'$.

It is possible to classify all boundary components of I in terms of a special subset of them:
A subset $\Theta \subset \Pi$ is called *pure* if either $\Theta = \emptyset$ or if all connected components of Θ (in an obvious sense) are of infinite type.

To any pure subset $\Theta \subset \Pi$ we may associate the following subset $I(\Theta)$ of I:

$$I(\Theta) = \{\omega \in I | \omega(h_i) = 0 \text{ for all } i \text{ such that } \alpha_i \in \Theta\}.$$

We now have the following result, essentially due to Looijenga ([6]):

Theorem:

i) Let $\Theta \subset \Pi$ be pure. Then $I(\Theta)$ is a rational boundary component of I.

ii) Let $I' \subset I$ be a boundary component. Then there is a unique pure $\Theta \subset \Pi$ and a $w \in W$ such that $I' = w.I(\Theta)$. In particular, all boundary components of I are rational.

Example: We take up the previous example. There are 3 pure subsets of Π:

$$\emptyset \quad , \qquad \Theta = \{\alpha_1, \alpha_2\} \qquad , \ \Pi = \{\alpha_1, \alpha_2, \alpha_3\}.$$

The corresponding boundary components are

$$I \quad , \quad \begin{array}{l}\text{all rational half–lines on}\\ \text{the positive light cone}\end{array} \quad , \quad \{0\}.$$

To determine the closure $\overline{T}$ of T in $\overline{G}$ we first have to describe the restriction of $\mathbb{C}[G]$ to T. Since all weights of a module $L(\Lambda)$, $\Lambda \in P^+$, are contained in $I \cap P$, and since $\overline{C} \cap P = P^+$ we obtain

$$\mathbb{C}[G]|_T = \mathbb{C}[P \cap I],$$

the semigroup algebra of $P \cap I$. It is easily seen that the induced topology on $\mathbb{C}[P \cap I]$ is discrete, thus

$$\overline{T} = \mathrm{Specm}^{\circ}\mathbb{C}[P \cap I] = \mathrm{Specm}\,\mathbb{C}[P \cap I].$$

Through $\mathbb{C}[P \cap I]$ is not finitely generated its maximal spectrum can be determined similarly as in the usual "finite type" theory of torus embeddings (cf. e.g. [1]), i.e. one has

$$\overline{T} = \dot{\bigcup_{I'}} T/\mathrm{Ann}(I') = \dot{\bigcup_{\Theta}} \bigcup_{w \in W} T/w\mathrm{Ann}(I(\Theta))w^{-1} \ ,$$

where $\mathrm{Ann}(I') = \{t \in T | \omega(t) = 1 \text{ for all } \omega \in I'\}$ and where I' (resp. Θ) runs through all rational boundary components of I (resp. all pure subsets of Π).

As a subset of M, the completion $\overline{T}$ has a quite natural representation theoretic realization:

Let $\Theta \subset \Pi$ be a pure subset. We define the pojection operator $e(\Theta) \in M$ by

$$e(\Theta)v = \begin{cases} v & \text{if} \quad v \in L(\Lambda)_\mu \quad \text{and} \quad \mu \in I(\Theta) \\ 0 & \text{if} \quad v \in L(\Lambda)_\mu \quad \text{and} \quad \mu \notin I(\Theta). \end{cases}$$

Then the boundary stratum $T/\mathrm{Ann}\,(I(\Theta))$ is realized as the T–orbit $T.e(\Theta)$ of $e(\Theta)$ under left multiplication by T. To realize $e(\Theta)$ as a boundary point of $\overline{T}$ choose a one–parameter subgroup $\gamma \in H = \mathrm{Hom}\,(\mathbb{C}^*, T)$ such that $\omega(\gamma) \geq 0$ for all $\omega \in I$ and $\omega(\gamma) = 0$ exactly when $\omega \in I(\Theta)$. Since for all $s \in \mathbb{C}^*$, $v \in L(\Lambda)_\omega$, we have

$$\gamma(s)v = s^{\omega(\gamma)}v \ ,$$

we clearly obtain (in M)

$$\lim_{s \to 0} \gamma(s) = e(\Theta) \ .$$

5. Unipotent subgroups

To study the unipotent radicals U^+, U^- of B^+, B^- as well as those of general parabolic subgroups we have to take a closer look at the action of G on $L(\Lambda), \Lambda \in P^+$. We consider $L(\Lambda)$ as a variety with the coordinate ring $\mathbb{C}[L(\Lambda)]$ generated by the functions $c_w : L(\Lambda) \to \mathbb{C}, c_w(v) = \langle v, w \rangle$, and equipped with the appropriate "cofinite" topology. Then, for any fixed $v \in L(\Lambda)$, the orbit map

$$\begin{array}{ccc} M & \longrightarrow & L(\Lambda) \\ m & \longmapsto & mv \end{array}$$

is a morphism of varieties (with continuous comorphism $\mathbb{C}[L(\Lambda)] \to \mathbb{C}[M]$). We shall make use of the following results of Kac and Peterson ([10],[3] Lemma 4.3)

- The Kostant cone $\mathcal{V}(\Lambda) = (Gv_0) \cup \{0\}$, with $v_0 \in L(\Lambda)_\Lambda \backslash \{0\}$, is Zariski closed in $L(\Lambda)$.
- If Λ is a regular dominant weight, $\Lambda \in P^{++}$ (i.e. $\Lambda(h_i) > 0$ for $i = 1, \dots, l$), then $\mathbb{C}[G]|_{U^-}$ is generated by the matrix coefficients c_{xv_0, v_0}, x running through all elements in $\mathbf{g}$ (in fact, $x \in \mathbf{g}^- = \bigoplus_{\alpha \in \sum^-} \mathbf{g}_\alpha$, where $\sum^-$ is the system of all negative roots, is sufficient).

Theorem ([8] Satz 5.6,1)): The groups U^+ und U^- are Zariski closed in M.

Proof: Because of the existence of the anti–involution $* : G \to G$ it is sufficient to consider U^-. Assume $v_0 \in L(\Lambda)_\Lambda \backslash \{0\}$ ($\Lambda \in P^{++}$) chosen such that $\langle v_0, v_0 \rangle = 1$. This implies

$$\begin{array}{lll} c_{v_0,v_0}(u) = 1 & \text{for all} & u \in U^-, \ \text{and} \\ c_{v_0,v_0}(\varphi) = 1 & \text{for all} & \varphi \in \overline{U^-}. \end{array}$$

Let $\varphi \in \overline{U^-}$. Then $\langle v_0, \varphi v_0 \rangle = 1$ implies $\varphi v_0 \neq 0$. Since $M \to L(\Lambda), m \mapsto m v_0$, is continuous and $\mathcal{V}(\Lambda)$ is closed in $L(\Lambda)$ we get $\varphi v_0 \in G v_0 \subset \mathcal{V}(\Lambda)$. Thus, using the Birkhoff decomposition of G, we find $u \in U^-, n \in N$ such that

$$\varphi v_0 = u^- n v_0 \ .$$

Because of $(u^-)^* \in U^+$ we have

$$1 = \langle v_0, \varphi v_0 \rangle = \langle (u^-)^* v_0, n v_0 \rangle = \langle v_0, n v_0 \rangle$$

and thus $n = 1$, or $\varphi v_0 = u^- v_0$. This implies $c_{x v_0, v_0}(\varphi) = c_{x v_0, v_0}(u^-)$ for all $x \in \mathbf{g}$, or $\varphi = u^- \in U^-$, q.e.d.

Recall that any subset $\Psi \in \Pi$ gives rise to a Weyl subgroup

$$W_\Psi = \langle s_{\alpha_i} | \alpha_i \in \Psi \rangle$$

and parabolic subgroups

$$P_\Psi^\pm = \langle B^\pm, W_\Psi \rangle$$

with unipotent radicals

$$U_\Psi^\pm = \bigcap_{w \in W_\Psi} w U^\pm w^{-1} \ .$$

It is obvious that $U_\Psi^\pm$ are Zariski closed in M, as well.

6. The main result

For any $i \in \{1, \ldots, l\}$ we fix a highest weight vector $v_i \in L(\Lambda_i)_{\Lambda_i} \backslash \{0\}$ and define the principal open subset $D_i \subset \overline{G}$ by

$$D_i = \{\varphi \in \overline{G} | c_{v_i,v_i}(\varphi) \neq 0\} .$$

We can almost cover $\overline{G}$ by these sets. Let $\Pi_\infty \subset \Pi$ the maximal pure subset of Π, i.e. Π is the "orthogonal" union of the set Π_∞ and a subset $\Pi \backslash \Pi_\infty$ of finite type.

Proposition A ([M], Satz 5.16): We have

$$\bigcup_{i=1}^{l} \bigcup_{g,h \in G} gD_ih = \overline{G} \backslash T.e(\Pi_\infty) .$$

Proof: To simplify our presentation, we shall assume $\Pi = \Pi_\infty$ and $P^\circ = \{0\}$. Then $e(\Pi) = e(\Pi_\infty) \in M$ is characterized by the property $e(\Pi)v = 0$, for all $v \in L(\Lambda), \Lambda \in P^+ \backslash \{0\}$. Consider $\varphi \in \overline{G}$ and assume $\varphi \notin gD_ih$ for all $i \in \{1, \ldots, l\}, g, h \in G$. Then

$$\langle gv_i, \varphi hv_i \rangle = 0, \quad \text{for all} \quad i, g, h .$$

Since $L(\Lambda_i)$ is spanned by all $gv_i, g \in G$, we obtain $\varphi|_{L(\Lambda_i)} = 0$. Since any $L(\Lambda), \Lambda \in P^+ \backslash \{0\}$ is made up from tensor products of the $L(\Lambda_i)$ and subsequent reduction, we get

$$\varphi|_{L(\Lambda)} = 0 \quad \text{for all} \quad \Lambda \in P^+ \backslash \{0\}$$

or $\varphi = e(\Pi)$.

(This proof can be easily adopted to the general case.)

As a next step, we shall determine the structure of the open sets $D_i \subset \overline{G}$. For that recall the parabolic subgroups

$$P_i^\pm = P_{\Pi \backslash \{\alpha_i\}}^\pm$$

with unipotent radicals

$$U_i^\pm = U_{\Pi \backslash \{\alpha_i\}}^\pm ,$$

Levi subgroup $G_i = P_i^+ \cap P_i^-$ and Weyl group $W_i = W_{\Pi\setminus\{\alpha_i\}}$. Then G_i is the Kac–Moody group attached to the realization $(H, \Pi\setminus\{\alpha_i\}, \Pi^\vee\setminus\{h_i\})$. Let $\mathbb{C}[G_i]$ denote the algebra of strongly regular functions on G_i and let $\mathbb{C}[G]_i$ denote the algebra of restricted functions from $\mathbb{C}[G]$ to the subgroup G_i. Then the function c_{v_i,v_i} restricts to the character Λ_i on G_i, and representation theoretic arguments quickly show (cf. [8], section 5.1.2):

Lemma: The inclusion $\mathbb{C}[G]_i \subset \mathbb{C}[G_i]$ induces an isomorphism from the localization of $\mathbb{C}[G]_i$ with respect to Λ_i to $\mathbb{C}[G_i]$:

$$(\mathbb{C}[G]_i)_{\Lambda_i} \xrightarrow{\sim} \mathbb{C}[G_i]\ .$$

Proposition B: For any $i \in \{1, \ldots, l\}$ we have an isomorphism of infinite–dimensional varieties

$$D_i = U_i^- \times \mathrm{Specm}^\circ \mathbb{C}[G_i] \times U_i^+\ .$$

Proof: Let us first look at $D_i \cap G$. Then the Birkhoff–decomposition

$$G = \bigcup_{w \in W} U^- wTU^+$$

gives

$$D_i \cap G = \bigcup_{w \in W_i} U^- wTU^+ = U_i^-.G_i.U_i^+ \ \ \text{(direct product)}\ .$$

Recall that the $U_i^\pm$ are closed in M, therefore in $\overline{G}$ and in D_i. By the Lemma, the closure of G_i in D_i can be identified with $\mathrm{Specm}^\circ\, \mathbb{C}[G_i]$. This gives the claim.

Applying downward induction to Propositions A and B we arrive at our main result.

Theorem([8], Satz 5.18): We have

$$\overline{G} = \{ge(\Theta)h | \Theta \subset \Pi \ \text{ pure } \ g, h \in G\} = G.\overline{T}.G\ .$$

Remarks: Proposition B for the case of the minimal parabolic B^+ may already be found in [3], Lemma 4.4. Its general version for arbitrary parabolics is due to Kashiwara ([5], Proposition 5.3.5), who has also given a form of Proposition A in a somewhat different context ([5], Proposition 6.3.1).

7. An application

In [8] one finds many more results on the structure of $\overline{G}$. Here, we want to conclude with an application to the adjoint quotient of G studied in [12], [13], [14] (details are forthcoming). Recall that G admits a "parabolic" partition

$$G = \bigcup_{\substack{\Theta \subset \Pi \\ \text{pure}}} G(\Theta)$$

parallel to a stratification of $\overline{T}/W$

$$\overline{T}/W = \bigcup_{\Theta \subset \Pi} (\overline{T}/W)(\Theta)$$

$$((\overline{T}/W)(\Theta) \text{ the image of } T/\operatorname{Ann}(I(\Theta)) \text{ in } \overline{T}/W).$$

The adjoint quotient defined in [12], [13] is a conjugation invariant map

$$\chi : G \to \overline{T}/W$$

mapping $G(\Theta)$ to $(\overline{T}/W)(\Theta)$ for any pure $\Theta \subset \Pi$. With the help of a theory of "optimal one–parameter semisubgroups" in $\overline{G}$ the partition and the map χ can be extended to a conjugation invariant map $\overline{\chi} : \overline{G} \to \overline{T}/W$ with the following properties, basic in geometric invariant theory:

- Every fibre of $\overline{\chi}$ contains a unique closed conjugacy class,
- two elements $\varphi, \psi \in \overline{G}$ are mapped to the same point in $\overline{T}/W$ if and only if the closures of their conjugacy classes meet,

$$\overline{(Ad(G)\varphi)} \cap \overline{(Ad(G)\psi)} \neq \emptyset \, .$$

Remarks: 1) If one considers $\chi : G \to \overline{T}/W$ these statements hold only for the "classical" part $G(\emptyset)$ mapping onto T/W.
2) The closed (= minimal = semisimple) orbits in all fibres of $\overline{\chi}$ are given as the orbits of the elements $t.e(\Theta)$, $\Theta \subset \Pi$ pure, $t \in T$.

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